AIAA JOURNAL
Vol. 33, No. 9, September 1995

Parameter Estimation for Weibull Probability Distribution Function of Initial Fatigue Quality

Andrea Pieracci*
University of Pisa, 56126 Pisa, Italy

This paper considers the problem of determining the parameters of a Weibull distribution when a limited set of experimental data of such a distribution is available. The problem is of particular interest in the probabilistic approach to durability analysis of flight structures, in which the distribution of time to crack initiation for a given crack length as well as the equivalent initial flaw size distribution are usually described through a three-parameter Weibull distribution. The aims of this paper are the determination of a high-quality estimation procedure and the solution of the problem of the determination of the number of experimental data that are necessary to estimate, with a given confidence level, the parameters of the distribution and some related statistics. The maximum likelihood and the least square methods are compared with regard to the estimation quality. It is shown, by Monte Carlo simulations, that the maximum likelihood approach gives better results than the least square approach. The problem of obtaining confidence intervals for the number of cracks with greater length than the reference one for a fixed service time on a single sample basis is addressed and solved through the bootstrap method.

Nomenclature

a_0 = reference crack length
$e(P)$ = relative error in probability for a fixed service time between actual and estimated Weibull distribution
$e(t)$ = relative error in time for a fixed probability level between actual and estimated Weibull distribution
$F_T(t)$ = cumulative probability function of crack exceedance
K = parameter related to the uniform distributed data set in the Monte Carlo simulation for the least square method
K' = parameter related to the uniform distributed data set in the Monte Carlo simulation for the maximum likelihood method
I = information matrix
n = number of sample data
P = probability of crack exceedance for the actual Weibull distribution function
$\hat{P}$ = estimated value of the probability of crack exceedance for the estimated Weibull distribution functions
P_i = random number generated with uniform distribution in the interval [0, 1]
$P(T < t)$ = cumulative probability function
p = probability of crack exceedance in the binomial distribution
T = time to crack initiation
t = service time
V^0 = dispersion matrix, i.e., inverse of the information matrix
α = shape parameter of the Weibull distribution function
$\hat{\alpha}$ = estimated shape parameter of the Weibull distribution function
$\bar{\alpha}$ = mean of the estimated shape parameters of the Weibull distribution function
β = scale parameter of the Weibull distribution function
$\hat{\beta}$ = estimated scale parameter of the Weibull distribution function
$\bar{\beta}$ = mean of the estimated scale parameters of the Weibull distribution function
γ = confidence coefficient
δ = confidence level
ε = lower bound parameter of the Weibull distribution function
ϑ_r = parameter of a multivariable distribution function
λ = parameter related to the uniform distributed data set in the Monte Carlo simulation for the least square method
λ' = parameter related to the uniform distributed data set in the Monte Carlo simulation for the maximum likelihood method
σ_N = standard deviation in the normal approximation of the binomial distribution
$\sigma_{\hat{\alpha}}$ = standard deviation of the estimation of the shape parameter of the Weibull distribution
$\sigma_{\hat{\beta}}$ = standard deviation of the estimation of the scale parameter of the Weibull distribution

Introduction

According to the probabilistic durability philosophy of the U.S. Air Force,[1–9] the analysis has to be based on two related probability distributions that characterize the initial fatigue quality of the component under examination and give the elements for evaluating the fatigue wear out development. Such distributions are the distribution of the time to crack initiation (TTCI) and the equivalent initial flaw size (EIFS) distribution, which are related to each other by the propagation curve for the component, by the load spectra, and by the stress level. The probability of having a greater crack length than the reference one at a given time t is given by the corresponding TTCI cumulative probability distribution[8]

$$F_T(t) = P[T \le t] = 1 - e^{-[(t-\varepsilon)/\beta]^\alpha} \tag{1}$$

In the present work it is supposed that the assumption of the lower bound of the distribution equal to zero could be sufficiently representative of a typical aircraft structure. This is equivalent to assuming that the probability of having a crack of the reference crack length may be very low but in any case nonzero at a time immediately after the airplane goes into service. Equation (1) becomes

$$F_T(t) = P[T \le t] = 1 - e^{-(t/\beta)^\alpha} \tag{2}$$

The determination of P for a given t as well as the determination of t for a given P are strongly dependent on the values of α and β. The procedure by which the parameters are obtained and the number of data used for this evaluation affect the estimation of the bound between P and t. According to the U.S. Air Force procedure, a least square (LS) approach has to be followed for the

Received July 18, 1994; revision received Dec. 28, 1994; accepted for publication Jan. 7, 1995.

*Assistant Professor, Dipartimento di Ingegneria Aerospaziale, Via Diotisalvi 2.

estimation of the parameters. No reference is made to the number of experimental data necessary to achieve some given confidence level. Subsequently it will be shown that the maximum likelihood (ML) estimation procedure provides more accurate values and that the accuracy of the estimate varies with the amplitude of data sample considered.

Least Squares Method

For the reference crack length a_0 the cumulative TTCI distribution is obtained by ranking the t_i values in ascending order[5,6,8] using

$$F_T(t_i) = [r/(n+1)] \tag{3}$$

where r is the rank of the data, n the total number of data available for the reference crack size a_0, and t_i the TTCI for the ith crack, i.e., the time at which such crack has reached the reference length a_0.

The values of the parameters may be obtained from the following relations[5]:

$$\hat{\alpha} = \left[n \sum_{i=1}^{n} \ln t_i \ln\{-\ln[1 - F_T(t_i)]\} - \left(\sum_{i=1}^{n} \ln t_i \right) \times \left(\sum_{i=1}^{n} \ln\{-\ln[1 - F_T(t_i)]\} \right) \right] \Bigg/ \left[n \sum_{i=1}^{n} \ln t_i^2 - \left(\sum_{i=1}^{n} \ln t_i \right)^2 \right] \tag{4}$$

$$\hat{\beta} = \exp\left[\left(\hat{\alpha} \sum_{i=1}^{n} \ln t_i - \sum_{i=1}^{n} \ln\{-\ln[1 - F_T(t_i)]\} \right) \Big/ \hat{\alpha} n \right] \tag{5}$$

Maximum Likelihood Method

With f the Weibull distribution function, i.e.,

$$f = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} \exp\left[-\left(\frac{x}{\beta}\right)^{\alpha} \right] \tag{6}$$

the joint probability density function for the n data sample becomes

$$f = \left(\frac{\alpha}{\beta^\alpha}\right)^n \prod_{i=1}^{n} \left[t_i^{\alpha-1} \exp\left(-\frac{t_i^\alpha}{\beta^\alpha} \right) \right] \tag{7}$$

so that the log-ML distribution function is

$$\ln f(t) = n \ln \alpha - n\alpha \ln \beta + (\alpha - 1) \sum_{i=1}^{n} \ln t_i - \frac{1}{\beta^\alpha} \sum_{i=1}^{n} t_i^\alpha \tag{8}$$

and the set of equations to be solved to estimate the parameters is given by

$$\frac{\partial \ln f}{\partial \alpha} = \frac{n}{\alpha} - n \ln \beta + \sum_{i=1}^{n} t_i - \sum_{i=1}^{n} \left(\frac{t_i}{\beta}\right)^{\alpha} \ln\left(\frac{t_i}{\beta}\right) \tag{9a}$$

$$\frac{\partial \ln f}{\partial \beta} = -\frac{n\alpha}{\beta} + \frac{\alpha}{\beta} \sum_{i=1}^{n} \left(\frac{t_i}{\beta}\right)^{\alpha} \tag{9b}$$

Equating Eq. (9b) to zero yields

$$\hat{\beta} = \left(\frac{1}{\hat{\alpha}} \sum_{i=1}^{n} t_i^{\hat{\alpha}} \right)^{1/\hat{\alpha}} \tag{10}$$

Equating Eq. (9a) to zero and using Eq. (10) gives

$$\left(\sum_{i=1}^{n} t_i^{\hat{\alpha}} \ln t_i \right) \Bigg/ \sum_{i=1}^{n} t_i^{\hat{\alpha}} - \frac{1}{\hat{\alpha}} = \frac{1}{n} \sum_{i=1}^{n} \ln t_i \tag{11}$$

where the carets refer to the values of the estimates of the parameters. Then the estimates of α and β are obtained from Eq. (11) and Eq. (10), respectively.

Relations and Statistics for the Parameter Estimation with Regards to Monte Carlo Simulations

The main objective of the present investigation is to evaluate the quality of the estimations of the distribution parameters by the aforementioned methods. The parameters of the distribution and their variance are firstly evaluated with the two procedures just described and then compared. To reach this goal, Monte Carlo simulations are carried out. A set of random numbers with uniform distribution are generated and then, given the α and β values, they are transformed in random numbers with the Weibull distribution. The parameters of the Weibull distribution are estimated on the basis of the generated random number data set and, thereafter, they are compared with the actual values (i.e., the values of α and β used to transform the uniform distributed random number in Weibull distributed random number). The Weibull distribution to which the generated random numbers belong will be referred to subsequently as actual Weibull distribution. This is due to the fact that the estimated value of the parameters are to be compared with the values of the parameters of such a distribution.

The corresponding Rao–Cramer bound values are estimated as reference values for the variance of the parameters. Two quantities are of interest. The first one is the relative error between the time predicted from the estimated distribution and the time predicted from the actual Weibull distribution for a given value of the probability of exceedance (e.g., $P = 0.05$, 0.5, and 0.95). The second one is the relative error in probability between the estimated distribution and the actual one for some fixed values of the time; e.g., for the values of the time corresponding to a probability of crack exceedance of 0.05, 0.5, and 0.95 for the actual Weibull distribution.

To keep the analysis sufficiently general, the simulations have to be run with the parameters α and β belonging to a bounded region where they are likely to lie. It is clear that the definition of the amplitude of the region as well as of the number of couples of α and β considered have a great influence on the calculation efforts and on their amplitude. To limit the number of simulations without losing the generality of the analysis, the study of the dependence of the estimated values of the parameters on the parameter values of the actual Weibull distribution is of a certain interest.

Least Squares Estimation with Reference to Monte Carlo Simulations

A set of random numbers is generated with uniform distribution in the interval [0, 1]. This set, indicated by $\{P_i\}$, corresponds to the values of the probabilities of crack exceedance for the rivet holes. The related set of time of crack initiation values $\{t_i\}$ is obtained by inversion of Eq. (2),

$$t_i = \beta[-\ln(1 - P_i)]^{1/\alpha} \tag{12}$$

To obtain the value of the estimated parameter α one has to substitute the expression in Eq. (12) for t_i in Eq. (4) (see Ref. 10),

$$\hat{\alpha} = \alpha \frac{n \sum_{i=1}^{n} \ln\{-\ln[1 - F_T(t_i)]\} \ln[-\ln(1 - P_i)]}{\sum_{i=1}^{n} n\{\ln[-\ln(1 - P_i)]\}^2 - \left\{ \sum_{i=1}^{n} \ln[-\ln(1 - P_i)] \right\}^2} - \frac{\left(\sum_{i=1}^{n} \ln\{-\ln[1 - F_T(t_i)]\} \right) \left(\sum_{i=1}^{n} \ln[-\ln(1 - P_i)] \right)}{\sum_{i=1}^{n} n\{\ln[-\ln(1 - P_i)]\}^2 - \left\{ \sum_{i=1}^{n} \ln[-\ln(1 - P_i)] \right\}^2} \tag{13}$$

or

$$\hat{\alpha} = K\alpha \tag{14}$$

where K depends only on the generated uniform distributed random data set (the $\{P_i\}$ sample), so that it is not dependent on the actual Weibull distribution function.

Analogously, for the parameter β one obtains from Eq. (5)

$$\hat{\beta} = \exp \frac{nK\alpha \ln\beta + K\sum_{i=1}^{n} \ln[-\ln(1-P_i)]}{nK\alpha} - \frac{\sum_{i=1}^{n} \ln\{-\ln[1-F_T(t_i)]\}}{nK\alpha} \tag{15}$$

that is,

$$\ln\hat{\beta} = \ln\beta + (\lambda/\alpha) \tag{16}$$

where λ depends only on the set of random numbers generated with uniform distribution and not on the actual Weibull distribution function.

Maximum Likelihood Estimation with Reference to Monte Carlo Simulations

Substituting the expression in Eq. (12) in Eq. (11) (see Pieracci[10]) results in

$$\left\{\left(\sum_{i=1}^{n}[-\ln(1-P_i)]^{\hat{\alpha}/\alpha}\ln(1-P_i)]\right)\Big/\sum_{i=1}^{n}[-\ln(1-P_i)]^{\hat{\alpha}/\alpha}\right\} - \frac{\alpha}{\hat{\alpha}} = \left[\left(\sum_{i=1}^{n}\ln[-\ln(1-P_i)]\right)\Big/ n\right] \tag{17}$$

which is an implicit equation in $\hat{\alpha}/\alpha$ that needs to be solved recursively. The result will then be

$$\hat{\alpha}/\alpha = K' \tag{18}$$

where K' depends only on the considered uniform random number set.

Substituting Eq. (18) in Eq. (10), one obtains

$$\hat{\beta} = \beta\left\{\frac{1}{n}\sum_{i=1}^{n}[-\ln(1-P_i)]^{K'}\right\}^{1/K'\alpha} = \beta\lambda'^{1/K'\alpha} \tag{19}$$

where λ' depends only on the uniform distributed data set.

Relative Error in Probability of Least Squares and Maximum Likelihood Estimations in Monte Carlo Simulations

Consider the statistics given by the following expression:

$$e(P) = (P - \hat{P})/P \tag{20}$$

where P is the probability of having a crack longer than the reference one at a given time for the actual Weibull distribution, and $\hat{P}$ is the value of the probability obtained from the estimated distribution at the same time. It will be shown that this quantity is independent of the Weibull random number generating function, so that for its study it is sufficient to execute a simulation with only a couple of (α, β) values for a given sample amplitude. Indeed, for a given probability value one obtains by inverting Eq. (2)

$$\hat{P} = 1 - \exp\left\{-\left(\frac{t}{\hat{\beta}}\right)^{\hat{\alpha}}\right\} = 1 - \exp\left\{-\left\{\frac{\beta[-\ln(1-P)]^{1/\alpha}}{\hat{\beta}}\right\}^{\hat{\alpha}}\right\} \tag{21}$$

From this and considering Eqs. (14) and (16) for the case of LS estimation, it follows that

$$e(P) = \frac{P-\hat{P}}{P} = \frac{P - 1 + \exp\{-\{[-\ln(1-P)]\exp -\lambda\}\}}{P} \tag{22}$$

Therefore, the quantity $e(P)$ is not dependent on the actual Weibull distribution function for the case of LS parameter estimation and depends only on the set of uniform random numbers.

Let us consider now the case of ML estimation; it follows from Eqs. (18) and (19) that

$$\hat{P} = 1 - \exp\left\{-\left\{\frac{[-\ln(1-P)]^{K'}}{\lambda'}\right\}\right\} \tag{23}$$

The quantity $e(P)$ is not dependent on the actual Weibull distribution function in this case as well.

Relative Error in Time of Least Squares and Maximum Likelihood Estimations in Monte Carlo Simulations

Let us consider now the quantity $e(t)$ given by the relation

$$e(t) = [t(a_0) - \hat{t}(a_0)]/t(a_0) \tag{24}$$

where the t and $\hat{t}$, or TTCI in U.S. Air Force nomenclature, are relative to the same probability of having a crack longer than the reference one (i.e., a_0) for the actual and the estimated Weibull distributions. Inverting Eq. (2), for the case of LS estimation, it follows that

$$\frac{t(a_0)}{\hat{t}(a_0)} = \frac{\beta[-\ln(1-P)]^{1/\alpha}}{\hat{\beta}[-\ln(1-P)]^{1/\hat{\alpha}}} = \exp\left(\frac{\lambda}{\alpha}\right)[-\ln(1-P)]^{(K-1)/K\alpha} \tag{25}$$

For the case of ML estimation, it is

$$\frac{t(a_0)}{\hat{t}(a_0)} = \frac{\beta[-\ln(1-P)]^{1/\alpha}}{\hat{\beta}[-\ln(1-P)]^{1/\hat{\alpha}}} = \lambda'^{\frac{1}{\alpha}}[-\ln(1-P)]^{(K'-1)/K'\alpha} \tag{26}$$

Considering Eqs. (25) and (26), it follows that the results of the simulations depend only on the uniformly distributed data set generated for the simulation and on the parameter α of the actual Weibull distribution function.

Determination of the Rao–Cramer Bound for Parameters of the Weibull Distribution

In the case of a multiparameter distribution the information matrix is given by[11]

$$I = -E\left\{\frac{\partial^2 \ln(\vartheta_1,\ldots,\vartheta_K; x_1,\ldots,x_n)}{\partial\vartheta_r\partial\vartheta_s}\right\} = -nE\left\{\frac{\partial^2 \ln f(X;\vartheta_1,\vartheta_2,\ldots,\vartheta_K)}{\partial\vartheta_r\partial\vartheta_s}\right\}, \quad r,s = 1,2,\ldots,K \tag{27}$$

where ϑ_r and ϑ_s are the parameters of the distribution and $x_1,\ldots,x_n$ the data on which their estimation is based on.

In this case the Rao–Cramer bound is expressed by[11]

$$\sigma^2_{\hat{\vartheta}_i} \geq V^0_{ii} \tag{28}$$

where

$$V^0 = [I(\vartheta)]^{-1} \tag{29}$$

The matrix V^0 for the two-parameter Weibull distribution is obtained by developing the calculations in Eq. (27) (see Pieracci[10]):

$$V^0 = \frac{\beta^2}{1.6449n}\begin{bmatrix} \dfrac{\alpha^2}{\beta^2} & \dfrac{0.4228}{\beta} \\ \dfrac{0.4228}{\beta} & \dfrac{1.8237}{\alpha^2} \end{bmatrix} \tag{30}$$

so that

$$\sigma^2_{\hat{\alpha}} \geq 0.6079\alpha^2/n \tag{31}$$

$$\sigma^2_{\hat{\beta}} \geq 1.1087\beta^2/n\alpha^2 \tag{32}$$

These relations are true for unbiased estimators although for biased ones additional terms have to be considered (see Sorensen[12]). The bias terms cannot be easily obtained in closed analytical form for the preceding cases, and so the unbiased formulation is considered as a reference expression.

Ratio of Variance of Estimated Parameters of Distribution with Rao–Cramer Bound in Monte Carlo Simulations

The Rao–Cramer bound is an appropriate bound for unbiased estimators. For low numbers of sampling data the estimators, as well as the ML ones, are usually biased; in any case, the ratio of the variance of the estimators to the Rao–Cramer bound may be a useful mean to evaluate the quality of the estimation.

The ratio of the sample variance to the Rao–Cramer bound is

$$\frac{s^2}{\sigma^2_{\hat{\alpha}}} = \frac{n}{1.6449(n-1)} \sum_{i=1}^{n} \left(\frac{\hat{\alpha}_i}{\alpha} - \frac{1}{n} \sum_{i=1}^{n} \frac{\hat{\alpha}_i}{\alpha} \right)$$

$$= \frac{n}{1.6449(n-1)} \sum_{i=1}^{n} \left(K_i - \frac{1}{n} \sum_{i=1}^{n} K_i \right) \tag{33}$$

for the case of LS estimation. The same expression is obtained for ML estimation by substitution of K' for K. From Eq. (33) it results that the ratio considered is independent of the actual Weibull distribution function.

In the same way, for parameter β of the Weibull distribution, the ratio is

$$\frac{s^2}{\sigma^2_{\hat{\beta}}} = \frac{n\alpha^2}{1.1087(n-1)} \sum_{i=1}^{n} \left(\frac{\hat{\beta}_i}{\beta} - \frac{1}{n} \sum_{i=1}^{n} \frac{\hat{\beta}_i}{\beta} \right)^2 \tag{34}$$

so that by substituting the relation between the actual and the estimated values of the parameters, the dependence of the ratio on the parameter α of the actual Weibull distribution function is the result for both LS and ML procedures.

Consequences and Applications of Obtained Relations

From the obtained relations it follows that some simplifications are possible for the execution of the Monte Carlo simulations. As far as the evaluation of the estimation procedures for the parameter α is concerned, it follows from Eq. (33) that the ratio of the variance of the estimated value with the Rao–Cramer bound are not dependent on the Weibull random numbers generating function, so that only a simulation for a given sample amplitude is necessary to compare the two estimation procedures. The ratio between the variance of the parameter β and the corresponding Rao–Cramer bound depends on α, so that in that case simulations with different α values of the generating function in the range of interest should be carried out. Regarding the quantities, $e(P)$ and $e(t)$, it has been previously determined that for the quantity $e(P)$ it is not necessary to identify a two-dimensional region where the coupled (α, β) plausibly lie, because of the independence of that quantity on the actual Weibull distribution function. For the second quantity, $e(t)$, it has been shown that it varies with α, so that it has to be evaluated in a plausible α range. These considerations allow a reduction of the number of simulations required to evaluate and compare the LS and ML estimation procedures without losing the generality of the results which will be obtained.

Some tests have been executed to evaluate the number of the Monte Carlo simulations required to achieve stability in the results. It has been determined that a number of 128,000 simulations may be sufficent for the aim of the present work. The results showed that the relative difference between the evaluations of the quantities of interest obtained by $1.28E+5$ and by $1.28E+6$ simulations was lower than 1%.

Analysis of Simulation Results

The first purpose of the analysis is to examine the quality of the parameter estimation. To reach this goal the sample variance is estimated by varying the number of the data per sample and then comparing it with the Rao–Cramer bound for the case of unbiased estimator. The results of the simulations are shown in Figs. 1 and 2. It is shown that the ML estimation is better than the LS estimation, at least for the Weibull random number generating function that has

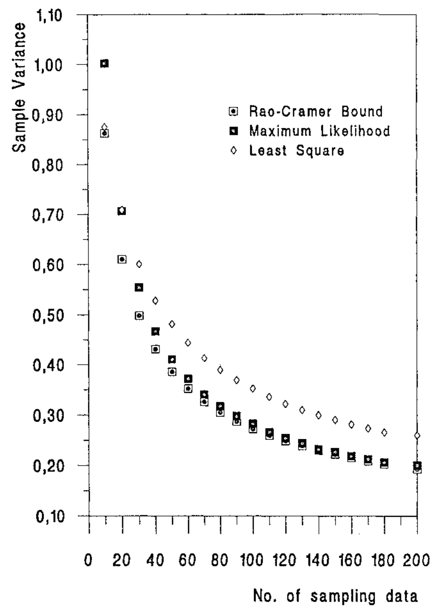

Fig. 1 Comparison of the variance of the parameter α of the Weibull distribution obtained by Monte Carlo simulation for ML and LS methods with the unbiased Rao–Cramer bound.

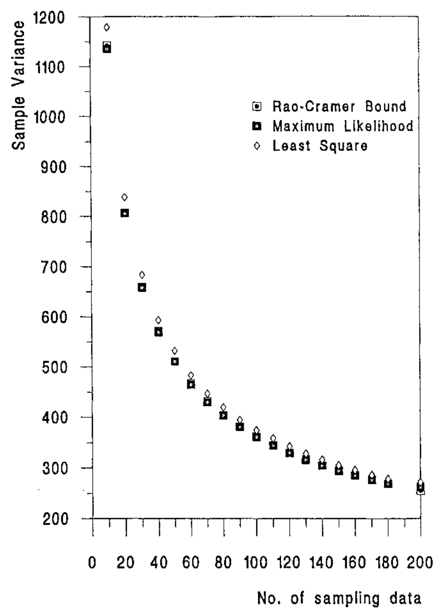

Fig. 2 Comparison of the variance of the parameter β of the Weibull distribution obtained by Monte Carlo simulation for ML and LS methods with the unbiased Rao–Cramer bound.

been simulated ($\alpha = 3.5$, $\beta = 12{,}000$). The LS approach, particularly for the β estimation, gives results very near to the Rao–Cramer bound. However, the dependence of the quantities in Fig. 2 on the shape parameter of the Weibull distribution used for the Monte Carlo simulations does not allow general considerations to be obtained. Some other simulations with α varying in a plausible range should be executed to do this.

In Figs. 3–5 the results of the simulation for the statistics $e(P)$ are reported for the case of LS and ML estimation varying the data sample amplitude. From the comparison the results show that the ML estimation is better than the LS one for the case of $P = 0.05$ and $P = 0.95$ in the actual distribution. The reverse is true for the case of $P = 0.5$. Some other simulations have been carried out which show that for values of probability of crack exceedance of about 0.5 (0.4–0.6) the LS method allows better estimates than the ML method.

These results may be explained in the following way. There will be a certain value of the time for which the estimated and the actual distribution functions will give the same value for the probability

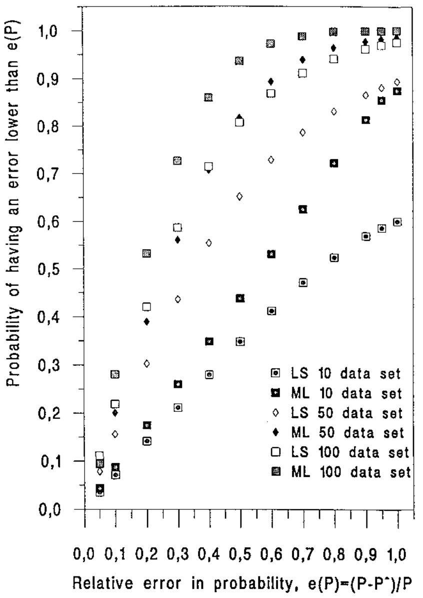

Fig. 3 Comparison of the probability functions of $e(P)$ relative to the time corresponding to $P = 0.05$ in the actual Weibull distribution function for LS and ML methods.

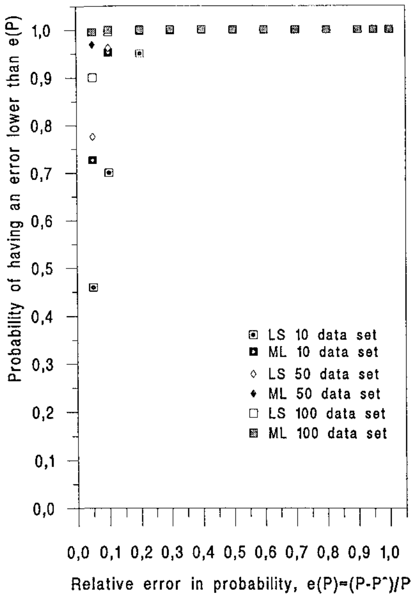

Fig. 5 Comparison of the probability functions of $e(P)$ relative to the time corresponding to $P = 0.95$ in the actual Weibull distribution function for LS and ML methods.

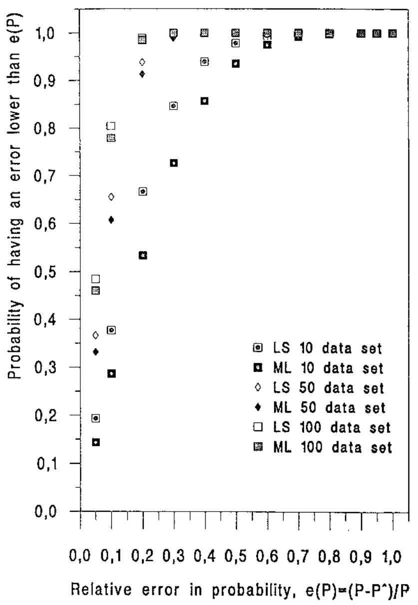

Fig. 4 Comparison of the probability functions of $e(P)$ relative to the time corresponding to $P = 0.5$ in the actual Weibull distribution function for LS and ML methods.

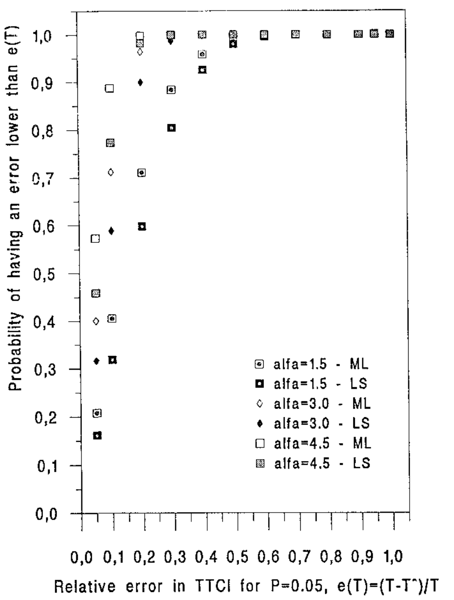

Fig. 6 Probability functions of $e(t)$ relative to crack exceedance probability $P = 0.05$ for the LS and the ML methods for different α values.

of crack exceedance. That value of the time may be obtained by equating the expressions of the probability of crack exceedance for the two distribution functions. The time at which the probability obtained by the two distributions is the same will not generally be a value corresponding to the tails of the distributions, an estimated value of which is more likely to be appreciably different from the real one. The equality is most likely to happen in the central part of the interval of variation of the probability ([0, 1]), i.e., far from the tails of the distribution function. The ML and LS estimated distributions will not generally give the same probability of crack exceedance as the actual distribution function at the same time. Thus, it will happen that, although the ML estimation is generally better than the LS estimation, there will be a certain interval of time values, and a corresponding interval of the probability of crack exceedance values, for which the LS approach will be better than the ML approach. This follows from the fact that the LS probability curve is going to intersect the actual probability curve, so that the corresponding error in probability goes to zero, for a value of the time at which this does not happen for the ML estimation. The estimate of the probability of crack exceedance becomes better when the value of the TTCI, at which the estimate is compared with the actual value, is relative to a higher probability of crack exceedance in the actual distribution. It has to be noted (but not for the case in Fig. 3) that the difference between the estimates of the two methods decreases when the data set is enlarged.

For the U.S. Air Force cases the parameter α is considered to lie in the interval 2.5–4.5. Simulations with three different α values have been carried out ($\alpha = 1.5, 3.0$, and 4.5), showing that the quantity $e(t)$ for a fixed probability level decreases when α grows.

Evaluations of the quantity $e(t)$ for the three α values for both ML and LS estimation are reported in Figs. 6–8 for 100 data samples, whereas the results obtained for $\alpha = 1.5$ with varying amplitude of the data samples are reported in Figs. 9–11. For the case of $P = 0.5$ (Fig. 10) the results of the two methods are almost equal. For the other cases, the ML estimation procedure gives better results, although the variance of the ML estimate is greater than the LS estimate for small data samples (see Fig. 1). The simulation results show that the ML estimation method has to be followed to obtain

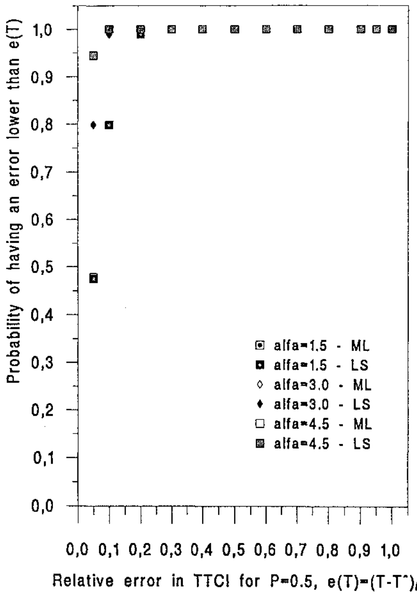

Fig. 7 Probability functions of $e(t)$ relative to crack exceedance probability $P = 0.5$ for the LS and the ML methods for different α values.

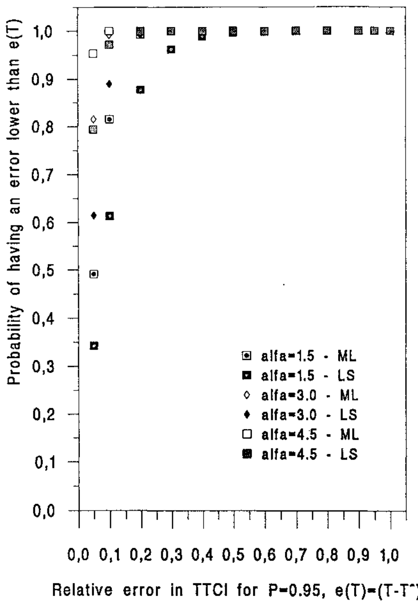

Fig. 8 Probability functions of $e(t)$ relative to crack exceedance probability $P = 0.95$ for the LS and the ML methods for different α values.

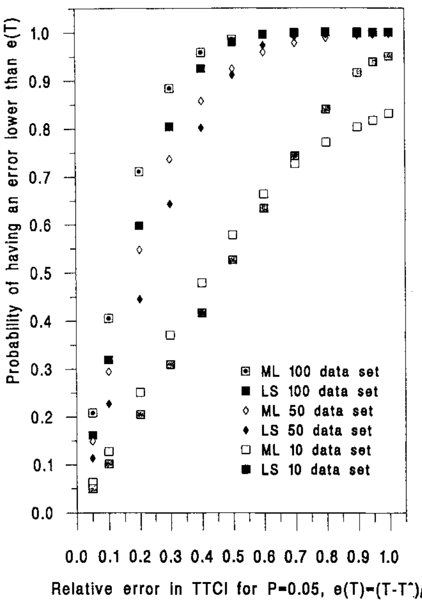

Fig. 9 Comparison of the probability functions of $e(t)$ relative to crack exceedance probability $P = 0.05$ ($\alpha = 1.5$) for the LS and ML methods for different data samples.

higher quality estimations. Attention must be paid to the fact that the data set has to be enlarged enough in order to achieve a high probability of well estimating the actual value of the parameters and of the probability of crack exceedance.

It has also to be noted that when the value of the probability of crack exceedance is varied, the number of data in the sampling set, which is necessary to achieve a certain confidence level in the statistics estimation, varies as well. The results of the simulations may be used as a tool to plan the number of experimental data and of the tests necessary to achieve the desired prediction quality.

The problem of estimating the variance of the parameter of the distribution and of the quantities of interest for a given data sample, e.g., for a given set of data that represents the results of an experimental test activity, remains to be solved. Such a problem is addressed in the following.

Confidence Interval Estimation for the Number of Cracks Longer than the Reference One

The ML and the LS estimation procedures allow point estimators of the parameters of the Weibull distribution to be obtained. In the approach proposed by the U.S. Air Force the confidence intervals of the estimated parameters as well as of the probability of crack exceedance are not evaluated. An assumption of a large number of independent cracks is made, so that the binomial distribution may be approximated by the normal one.

Two facts have to be considered: the number of cracks is not always large and the point estimate of the probability of crack exceedance p is not exact. The first problem may be solved finding an interval (k_1, k_2) of minimum size, such that

$$\gamma = P\{k_1 \leq k \leq k_2\} = \sum_{k=k_1}^{k_2} \binom{k}{n} p^k q^{n-k} = 1 - \delta \tag{35}$$

or, solving the problem in a simpler way, searching the values of (k_1, k_2) that have the minimum size of the following intervals (see Papoulis[13]):

$$\sum_{k=0}^{k_1} \binom{n}{k} p^k q^{n-k} \leq \frac{\delta}{2} \qquad \sum_{k=k_2}^{n} \binom{n}{k} p^k q^{n-k} \leq \frac{\delta}{2} \tag{36}$$

For the second problem a Bayesian approach may be followed. Let us call A the event of having rivets with cracks greater than the reference crack length at a given time and p the value of the random variable corresponding to the probability of crack exceedance at that time, it is

$$\gamma = P(A) = \int_0^1 P(A|p) f(p) \, \mathrm{d}p \tag{37}$$

and, with the simplification of Eq. (36)

$$\begin{aligned} \sum_{k=0}^{k_1} \binom{n}{k} \int_0^1 p^k q^{n-k} f(p) \, \mathrm{d}p \leq \frac{\delta}{2} \\ \sum_{k=k_1}^{n} \binom{n}{k} \int_0^1 p^k q^{n-k} f(p) \, \mathrm{d}p \leq \frac{\delta}{2} \end{aligned} \tag{38}$$

To solve the problem in the way proposed in Eqs. (37) and (38), the probability distribution function must be known. The problem of the determination of the distribution function of the probability of crack exceedance at a fixed time and then the determination of the minimum interval (k_1, k_2) for a given confidence level may be

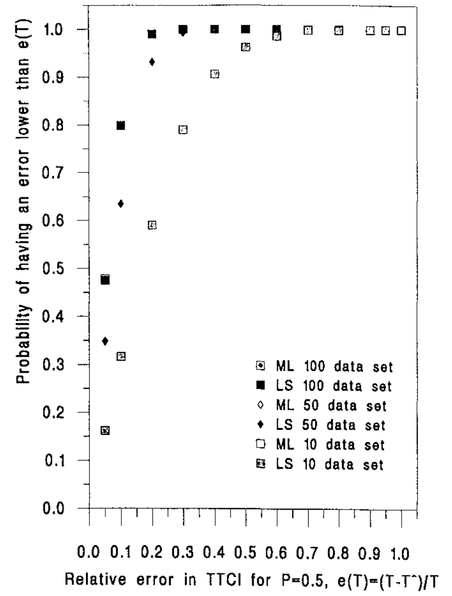

Fig. 10 Comparison of the probability functions of $e(t)$ relative to crack exceedance probability $P = 0.5$ ($\alpha = 1.5$) for LS and ML methods for different data samples.

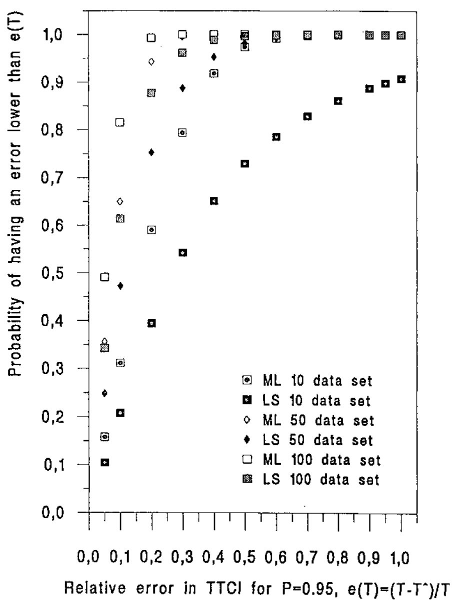

Fig. 11 Comparison of the probability functions of $e(t)$ relative to crack exceedance probability $P = 0.95$ ($\alpha = 1.5$) for LS and ML methods for different data samples.

solved through Monte Carlo simulation, using the so-called bootstrap method, which will be described and then applied in next.

Bootstrap Method

The bootstrap method[14–16] is a Monte Carlo procedure for obtaining standard error and confidence intervals for a parameter when a single data set is available. The bootstrap method estimates, through the actual data set, the discrete probability function to which the data belong, and uses such a function to carry out a Monte Carlo simulation with a generating replacement procedure to estimate the parameter of interest. The procedure can be applied to obtain confidence intervals by using a so-called percentile method,[15] based on the determination of the percentiles of the Monte Carlo simulated distribution of the parameter of interest. In the same way, one can evaluate the probability distribution function of the parameters of interest through a discrete approximation. This method has been used in the present work to obtain the approximate probability distribution of the probability of crack exceedance at a given time. This is done in order to evaluate the confidence interval given by Eq. (38)

Table 1 Confidence interval relative to ML point estimate N_{ML} and bootstrap interval estimate N_{boot} for a set of 30 cracks

	$T, P = 0.05$	$T, P = 0.5$	$T, P = 0.95$
N_{ML}	0–2	8–17	24–29
N_{boot}	0–3	7–18	23–29

Table 2 Confidence interval relative to ML point estimate N_{ML} and bootstrap interval estimate N_{boot} for a set of 50 cracks

	$T, P = 0.05$	$T, P = 0.5$	$T, P = 0.95$
N_{ML}	0–4	15–27	42–48
N_{boot}	0–4	13–30	41–49

Table 3 Confidence interval relative to ML point estimate N_{ML} and bootstrap interval estimate N_{boot} for a set of 100 cracks

	$T, P = 0.05$	$T, P = 0.5$	$T, P = 0.95$
N_{ML}	0–7	35–51	86–95
N_{boot}	0–9	29–58	83–97

for some reference cases developed to show the applicability of the proposed approach.

Application of the Bootstrap Method

The bootstrap method as described was used to evaluate the confidence intervals of the probability of crack exceedance corresponding to the time for which in the actual distribution such a probability has the values 0.05, 0.5, and 0.95. It is clear, from earlier discussions, that there is a difference in the confidence intervals for the number of cracks longer than a reference one which are obtained for the two cases of p exactly known and p known in a certain confidence interval.

An example was carried out to show such a difference. A random sample containing 30 data points was generated, point estimate of the parameters of the Weibull distribution were obtained with the ML approach, and then the bootstrap procedure was applied. It was shown that a number of simulations of the order of 1000 was sufficent to achieve the stability of the bootstrap results. The confidence intervals for the confidence level of 90% for the value of the probability of crack exceedance obtained with the bootstrap method were compared with the results which are obtained with the point estimate approach. For the point estimate, Eq. (36) was used instead of the normal approximation to allow a correct comparison. The results obtained with the two methods for estimating the behaviour of three different data samples (corresponding to 30, 50, and 100 rivets sets) are shown in Tables 1–3. It has to be noted that the error which is induced by the assumption of having the value of p exactly known may be not negligible. In any case, we have to consider that the bootstrap method provides correct confidence intervals, but always on the basis of the available information, i.e., of the experimental data set.

Conclusions

In the present paper ML and LS estimation approaches were compared by Monte Carlo simulations to evaluate which is the best procedure to estimate the parameters of a Weibull distribution modeling the initial fatigue quality of an aircraft structure. It was shown that the ML approach is generally better than the LS approach. It has been shown that the data sample has to be taken large enough so that accurate point estimates can be obtained. The generality of the results was guaranteed by the demonstration of the independence of the simulation results for the quantities of the actual Weibull distribution function. The results of the Monte Carlo simulations allow also the evaluation of the number of experimental data that are necessary to obtain the point estimate of the parameters and of the quantities of interest with a requested confidence level.

In the probabilistic approach proposed by the U.S. Air Force, point estimates were used to obtain probability of crack exceedance without considering that the estimated parameters of the distribution would lie in some specific confidence interval. The bootstrap procedure was shown to be useful to obtain the correct confidence interval from a single sample of data. The difference between the results of the estimations obtained using point estimations and the herein proposed approach may be not negligible, and so a more accurate bootstrap approach is suggested.

Acknowledgment

The work was financially supported by the National Research Council of the Republic of Italy (C.N.R.) with Grant 92.09150.PF74 in the framework of Project PFT2.

References

[1]Manning, S. D., and Yang, J. N., "Durability Methods Development—Phase I Summary," Air Force Wright Aeronautical Labs., AFWAL-TR-83-3027, Vol. 1, Wright–Patterson AFB, OH, 1984.

[2]Manning, S. D., and Yang, J. N., "Durability Analysis: State of the Art Assesment," Air Force Wright Aeronautical Labs., AFWAL-TR-83-3027, Vol. 2, Wright–Patterson AFB, OH, 1984.

[3]Manning, S. D., and Yang, J. N., "Structural Durability Survey: State of the Art Assesment," Air Force Wright Aeronautical Labs., AFWAL-TR-83-3027, Vol. 3, Wright–Patterson AFB, OH.

[4]Shinozuka, M., "Initial Quality Representation,"Air Force Wright Aeronautical Labs., AFWAL-TR-83-3027, Vol. 4, Wright–Patterson AFB, OH, 1984.

[5]Manning, S. D., and Yang, J. N., "Durability Analysis Methodology Development," Air Force Wright Aeronautical Labs., AFWAL-TR-83-3027, Vol. 5, Wright–Patterson AFB, OH, 1984.

[6]Manning, S. D., and Yang, J. N., "Durability Analysis Methodology Development—Phase II Documentation," Air Force Wright Aeronautical Labs., AFWAL-TR-83-3027, Vol. 7, Wright–Patterson AFB, OH, 1984.

[7]Rudd, J. L., Manning, S. D., Yang, J. N., and Yee, B. G. W., "Probabilistic Fracture Mechanics Analysis Methods for Structural Durability," *Proceedings of the AGARD Meeting on Behaviour of Short Cracks in Airframe Components*, Toronto, ON, Canada, Sept. 1982, pp. 20, 21.

[8]Manning, S. D., and Yang, J. N., "USAF Durability Design Handbook: Guidelines for the Analysis of Durable Aircraft Structures," Air Force Wright Aeronautical Labs., AFWAL-TR-83-3027, Wright–Patterson AFB, OH, 1984.

[9]Manning, S. D., and Yang, J. N., "USAF Durability Design Handbook: Guidelines for the Analysis of Durable Aircraft Structures," Air Force Wright Aeronautical Labs., AFWAL-TR-88-3119, Wright–Patterson AFB, OH, 1989.

[10]Pieracci, A., "Comparison Between ML and LS Parameter Estimation for Weibull Distribution of Initial Fatigue Quality," Rept. of the Dept. of Aerospace Engineering, Univ. of Pisa, A.D.I.A. 94-3, Pisa, Italy, June 1994.

[11]Lloyd, E., "Estimation: Introduction Survey," *Handbook of Applied Mathematics, Statistics*, Vol. 6, Pt. A, edited by E. Lloyd and W. Ledermann, Wiley, New York, 1984, pp. 111–114.

[12]Sorensen, H., W., *Parameter Estimation*, Marcel Dekker, New York, 1980, p. 382.

[13]Papoulis, A., *Probability and Statistics*, Prentice–Hall, Englewood Cliffs, NJ, 1990, p. 454.

[14]Efron, B., and Tibshirani, R., "Bootstrap Methods for Standard Errors, Confidence Intervals, and Other Measures of Statistical Accuracy," *Statistical Science*, Vol. 1, No. 1, 1986, pp. 54–77.

[15]Efron, B., "Bootstrap Confidence Intervals for a Class of Parametric Problems," *Biometrika*, Vol. 72, No. 1, 1985, pp. 45–58.

[16]Efron, B., "Nonparametric Estimates of Standard Error: the Jackknife, the Bootstrap, and other Resampling Methods," *Biometrika*, Vol. 68, 1981, pp. 589–599.